Analysis IV

(for EL, GM, MX)

Week 2 - 24.02.25

Recall: A function $f: \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_0 \in \mathbb{C}$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is finite.

Analogous to the definition of the derivative of a real function in Analysis I & II.

Cauchy-Riemann equations

Let $W \subseteq \mathbb{C}$ be an open set and $f : W \longrightarrow \mathbb{C}$,

$$z = x + yi \longrightarrow f(x+yi) = u(x,y) + v(x,y)\,i$$

be a <u>holomorphic</u> function. Then,

$$u, v \in C^1(W) \quad \text{and} \quad \boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}} \tag{*}$$

differentiable with continuous derivatives

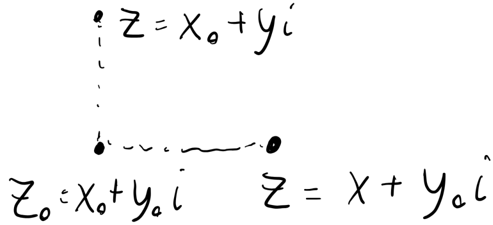
Conversely, if (*) are satisfied, then f is holomorphic.

We will prove the direct implication (the converse is more difficult).

If f is holomorphic on W, then $\forall z_0 \in W$:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

We will calculate the limit in two ways:



- Approaching z_0 from the x-direction:

$z_0 = x_0 + y_0 i$, $\quad z = x + y_0 i$

$$f'(z_0) = \lim_{x \to x_0} \frac{f(x + y_0 i) - f(x_0 + y_0 i)}{x + y_0 i - (x_0 + y_0 i)} =$$

$$= \lim_{x \to x_0} \frac{u(x, y_0) + v(x, y_0) i - u(x_0, y_0) - v(x_0, y_0) i}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} \cdot i$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \cdot i \qquad (1)$$

- Approaching $z_0 = x_0 + y_0 i$ from the y direction: $z = x_0 + yi$

$$f'(z_0) = \lim_{y \to y_0} \frac{f(x_0 + yi) - f(x_0 + y_0 i)}{\cancel{x_0} + yi - \cancel{x_0} - y_0 i}$$

$$= \lim_{y \to y_0} \frac{u(x_0, y) + v(x_0, y) \cdot i - u(x_0, y_0) - v(x_0, y_0) i}{(y - y_0) \cdot i}$$

$$= \frac{1}{i} \lim_{y \to y_0} \left(\frac{u(x_0, y) - u(x_0, y_0)}{y - y_0} + \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} i \right)$$

$$= \frac{1}{i} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \cdot i \right)$$

$$= \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} i \qquad (2)$$

$$① = ② \implies \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases} \qquad \blacksquare$$

Also from this proof:

Expressions for $f'(z_0)$:

$$f' = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \cdot i = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot i$$

Ex 1. $f(z) = z^2 = (x+yi)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2xy}_{v(x,y)} i$

Is it holomorphic over $\mathbb{C}$?

We check the Cauchy-Riemann eqns:

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

C-R are satisfied ✓

And

$$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = 2x + 2yi = 2z$$

Ex 2. The complex conjugate:

$$f(z) = \bar{z} = x - yi$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

$$\partial_x u \neq \partial_y v$$

$\Rightarrow$ Not holomorphic!

Another way of seeing the same fact:

If we try to compute the limit

$$\lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0}: \quad \bullet \text{ For } \begin{array}{l} z = x + y_0 i, \\ z_0 = x_0 + y_0 i \end{array}$$

$$= \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = +1$$

$\bullet$ For $z = x_0 + yi$:

$$= \lim_{y \to y_0} \frac{-yi + y_0 i}{yi - y_0 i} = -1 \neq +1$$

Ex 3 $f: \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$, $f(z) = \frac{1}{z}$

We have: $f(x+yi) = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$

So:

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

So C-R eqn: Satisfied.

And $$f' = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \cdot i$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} i$$

This is $= -\frac{1}{z^2}$

Indeed: $-\frac{1}{z^2} = -\frac{\bar{z}^2}{|z|^4} = -\frac{(x-yi)^2}{(x^2+y^2)^2} = \ldots$

Ex 4.

$$f(z) = e^z = e^{x+yi} = e^x \cdot e^{yi}$$

$$= e^x \cos y + e^x \sin y \cdot i$$

So: $\dfrac{\partial u}{\partial x} = e^x \cos y \qquad \dfrac{\partial v}{\partial x} = e^x \sin y$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

So C-R eqns: Satisfied.

And $f' = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x} i = e^x \cos y + e^x \sin y \cdot i$

$$= e^z.$$

Rules of differentiation for complex derivatives.

Let $f, g : W \longrightarrow \mathbb{C}$ be holomorphic.

Then:
$$(f+g)' = f' + g'$$
$$(f \cdot g)' = f' \cdot g + f \cdot g'$$
$$\left(f/g\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \quad \text{when } g \neq 0.$$

If the image of g is inside W:
$$(f \circ g)'(z) = f'(g(z)) \cdot g'(z).$$

These relations can be proved in exactly the same way as in the real case:

E.g.

$$(f \cdot g)'(z_0) = \lim_{z \to z_0} \frac{f(z) \cdot g(z) - f(z_0) g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{f(z) \cdot g(z) - f(z_0) \cdot g(z) + f(z_0) \cdot g(z) - f(z_0) g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \cdot g(z) + f(z_0) \cdot \frac{g(z) - g(z_0)}{z - z_0} \right)$$

$$= f'(z_0) \cdot g(z_0) + f(z_0) \cdot g'(z_0).$$

With these rules and the examples we computed earlier: It follows that the usual formulas for polynomials, trigonometric functions etc. hold.

E.g $\quad (z^k)' = k \cdot z^{k-1}, \quad k \in \mathbb{Z}$

$$(\cos(z))' = -\sin(z).$$

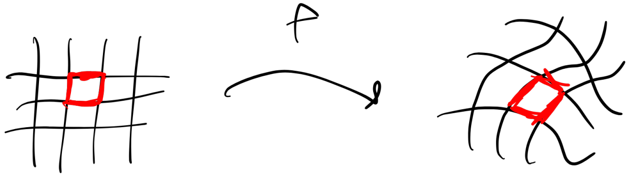
Aside: <u>A geometric property of holomorphic functions.</u>

If $f: \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic:

When viewed as a map $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, the Jacobian matrix $Df = \begin{pmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{pmatrix}$

tells us how f acts on infinitesimal

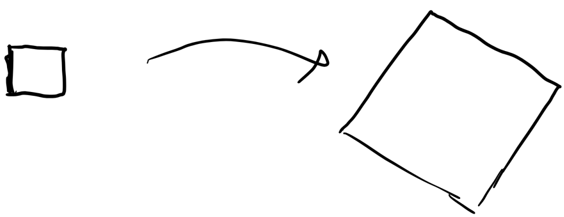
rectangles:



Cauchy-Riemann equations:

$\Rightarrow$ $Df = \lambda \cdot \text{Rotation}$

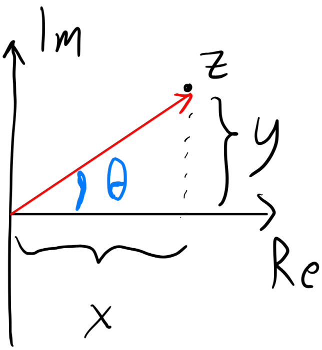
So infinitesimal square is mapped to square (no uneven stretching)



The complex logarithm

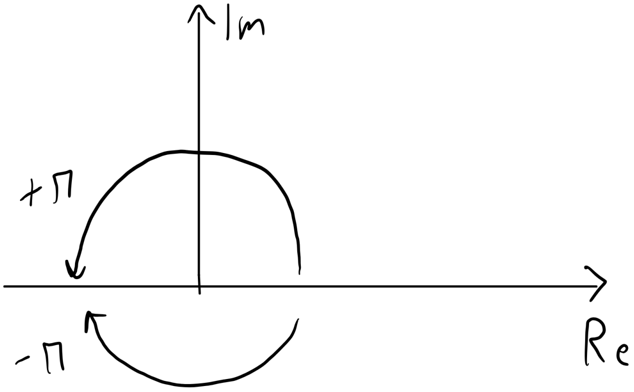
Recall: The principal argument of a complex number $z \neq 0$:

$$-\pi < \operatorname{Arg}(z) \leq \pi.$$



How to compute:

$$\operatorname{Arg}(z) = \begin{cases} \operatorname{Arctan}\left(\frac{y}{x}\right), & x > 0 \\ +\frac{\pi}{2}, & x = 0,\ y > 0 \\ \pi + \operatorname{Arctan}\left(\frac{y}{x}\right), & x < 0,\ y \geq 0 \\ -\frac{\pi}{2}, & x = 0,\ y < 0 \\ -\pi + \operatorname{Arctan}\left(\frac{y}{x}\right), & x < 0,\ y < 0 \end{cases}$$



Defined for $z \neq 0$, discontinuous on the negative real axis

How would we define the complex logarithm?

If $\log(z) = x + yi$, then we want

$$z = e^{x+yi}$$

Then: $z = |z| \cdot e^{i\,Arg(z)} = e^{x+yi} = e^{x} e^{yi}$

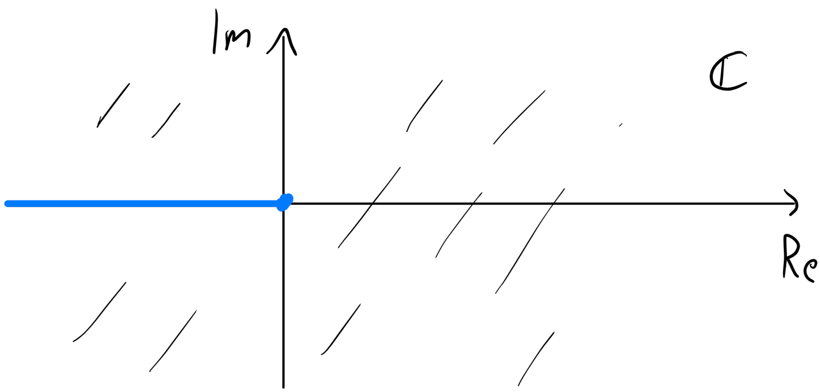
So: We define

$$\boxed{\log z = \log|z| + Arg\, z \cdot i}$$

- $\log z$ <u>not</u> defined for $z=0$.
- $\log z$: Discontinuous on the negative real axis.

But: $\log z$ is continuous on

$$\mathbb{C} \setminus \{ z \in \mathbb{C} : \operatorname{Im}(z) = 0, \operatorname{Re}(z) \leq 0 \}$$



The negative real axis: Branch cut

(Had we defined $\operatorname{Arg}(z)$ differently, we would have a different branch cut).

Real and imaginary parts:

$$\log z = \log|z| + \mathrm{Arg}(z) \cdot i$$

If $z = x + yi$: If $x > 0$,

$$\log z = \underbrace{\log\sqrt{x^2+y^2}}_{u(x,y)} + \underbrace{\mathrm{Arctan}\left(\frac{y}{x}\right)}_{v(x,y)} \cdot i$$

Easy to verify:

$\log z$ is holomorphic on

$$\mathbb{C} \setminus \{z : \mathrm{Im} z = 0, \ \mathrm{Re} z \leq 0\}$$

with $(\log z)' = \frac{1}{z}$.

Remark. For positive real numbers:

$$\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2)$$

No longer true for general complex numbers (due to discontinuity):

If $z_1 = z_2 = -1$, $\log z_1 = \log z_2 = i\pi$

$$\log(z_1 \cdot z_2) = 0.$$

But: It is true up to $2\pi i\mathbb{Z}$.

Power function with complex exponent.

Let $\gamma \in \mathbb{C}$. Define

$$f(z) = z^{\gamma} = e^{\log(z^{\gamma})} = e^{\gamma \cdot \log(z)}$$

$$= e^{\gamma \cdot \log|z| + \gamma \cdot \operatorname{Arg} z \cdot i}$$

Holomorphic over $\mathbb{C} \setminus \{z : \operatorname{Im} z = 0, \operatorname{Re} z \le 0\}$.

(as the composition of $e^{\gamma z}$ with $\log z$)

When $\gamma \in \mathbb{Z}$: Continuous (in fact, holomorphic) for $z \neq 0$.